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ℓ-Parametric eigenproblem in max-algebra

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Abstract

Denote $a \oplus b = \max(a, b)$, and $a \otimes b = a + b$ for $a, b \in R$ and extend this pair of operations to matrices and vectors in the same way as in conventional linear algebra, that is, if $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ are real matrices or vectors of compatible sizes then $C = A \otimes B$ if $c_{ij} = \Sigma_k^{\oplus} a_{ik} \otimes b_{kj}$ for all i, j. The symbol $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ denotes the matrix D with diagonal elements equal to $d_1, d_2, \ldots, d_n$ and off-diagonal elements equal to $-\infty$. For an arbitrary parameter $\varepsilon \in R$ and given square matrices $A = (a_{ij})$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_\ell, 0, \ldots, 0)$, $d_j = \varepsilon$, $1 \leqslant j \leqslant \ell$, we study the ℓ-*parametric eigenproblem*, i.e. problem of finding all $x_\varepsilon = (x_1(\varepsilon), x_2(\varepsilon), \ldots, x_n(\varepsilon))$ and $\lambda_{\varepsilon\ell}$, satisfying

$$A_{\varepsilon\ell} \otimes x_\varepsilon = \lambda_{\varepsilon\ell} \otimes x_\varepsilon,$$

where $A_{\varepsilon\ell} = A \otimes D$. We introduce some properties of general ℓ-*parametric eigenproblem* and the $O(n^3)$ algorithm which gives all solutions of the 1-*parametric eigenproblem* with respect to ε.

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1. Introduction

Discrete-event systems can represent a great number of systems in which the individual components move from event to event rather than varying continuously through time. A characteristic of many such discrete-event systems is that any given component must wait

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before proceeding to its next event until certain others have completed their current events. Specifically, significant effort has been developed to build up a theory similar to that of linear algebra, for instance to study systems of linear equations and an eigenproblem. Cuninghame-Green [5] discussed the following eigenproblem in connection with a hypothetical industrial discrete-event system—on an assembly line a man cannot begin a new assembly until, say, two inter-locking subassemblies have arrived from different sources with independent production rates. A natural way of describing such a system is to label the machines, e.g. $1, 2, \ldots, n$, and to describe the interferences by recurrence relations

$$x_i(r+1) = \max(x_1(r) + a_{1i},\ x_2(r) + a_{2i}, \ldots, x_n(r) + a_{ni}), \quad i \in \{1, 2, \ldots, n\}$$

and for $n = 3$ we have

$$x_3(r+1) = \max(x_1(r) + a_{13}, x_2(r) + a_{23}).$$

The last formula expresses the fact that machine 3 must wait to begin its $r+1$st cycle until machines 1 and 2 have both finished their rth cycle, the symbol $x_i(r)$ denoting the starting time of the rth cycle of machine i, and a_{ij} is corresponding activity duration.

Denote $a \oplus b = \max(a, b)$, and $a \otimes b = a + b$ for $a, b \in R$ and extend this pair of operations to matrices and vectors in the same way as in conventional linear algebra, that is, if $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ are real matrices or vectors of compatible sizes, then $C = A \otimes B$ if $c_{ij} = \Sigma_k^{\oplus} a_{ik} \otimes b_{kj}$ for all i, j.

By generalization of the discrete-event system (eigenproblem) mentioned above and using the above notation we obtain a description of the stable system (for some constant λ, the interval between the beginning of consecutive cycles on every machine has to be λ, i.e. $x(r+1) = A(r) \otimes x(r)$ and $x(r+1) = \lambda \otimes x(r)$) in vector–matrix notation as follows:

$$A \otimes x(r) = \lambda \otimes x(r).$$

The aim of this paper is to analyze this model if changes of activity durations a_{jk} for all j and $1 \leqslant k \leqslant \ell$, say, on value $a_{jk} + \varepsilon$ can occur.

2. Definitions and preliminary results

Let $G = (G, \otimes, \leqslant)$ be a linearly ordered, commutative group with neutral element $e = 0$. We suppose that G is radicable, i.e. for every integer $t \geqslant 1$ and for every $a \in G$, there exists a (unique) element $b \in G$ such that $b^t = a$. We denote $b = a^{1/t}$.

Throughout the paper $n \geqslant 1$, $m \geqslant 1$ are given integers. The set of $n \times m$ matrices over G is denoted by $G(n, m)$. We introduce further a binary operation $\oplus$ on G by the formula

$$a \oplus b = \max(a, b) \quad \text{for all } a, b \in G.$$

The triple $(G, \oplus, \otimes)$ is called *max-algebra*. If $G = (G, \otimes, \leqslant)$ is an additive group of real numbers, then $(G, \oplus, \otimes)$ is called *max-plus algebra* (often used in applications). The operations $\oplus, \otimes$ are extended to the matrix–vector algebra over G by the direct analogy to the conventional linear algebra. We extend G by a new element $-\infty$, we denote $G \cup \{-\infty\}$ by $\bar{G}$ and extend $\otimes$ and $\leqslant$ to $\bar{G}$: $a \otimes -\infty = -\infty \otimes a = -\infty$ and $-\infty < a$ for all $a \in$

G. The symbol $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ denotes the matrix D with diagonal elements equal to $d_1, d_2, \ldots, d_n$ and off-diagonal elements equal to $-\infty$. This matrix D will be called *diagonal* if all $d_1, d_2, \ldots, d_n \in G$. If $D = \mathrm{diag}(d_1, d_2, \ldots, d_\ell, e, \ldots, e)$, $d_j = \varepsilon$, $1 \leqslant j \leqslant \ell$ and $A \in G(n, n)$ denote $A_{\varepsilon^\ell} = A \otimes D$.

The aim of this paper is to present a description of the eigenvalues and to analyze the eigenspace with respect to ε. Below, we summarize and recall some of the main results. First, we introduce the necessary notation.

Let $N = \{1, 2, \ldots, n\}$ and let C_n be the set of all cyclic permutations defined on nonempty subsets of N. For a cyclic permutation $\sigma = (i_1, i_2, \ldots, i_l) \in C_n$ and for $A \in G(n, n)$, we denote l, the length of σ by $l(\sigma)$ and define

$$w_A(\sigma) = a_{i_1 i_2} \otimes a_{i_2 i_3} \otimes \cdots \otimes a_{i_l i_1}, \quad \mu_A(\sigma) = w_A(\sigma)^{1/l(\sigma)},$$

$$\lambda(A) = \sum_{\sigma \in C_n}^{\oplus} \mu_A(\sigma),$$

where $\sum^{\oplus}$ denotes the iterated use of the operation $\oplus$.

The *eigenproblem* in max-algebra is formulated as follows: given $A \in G(n, n)$, find $x \in G(n, 1)$ and $\lambda(A) \in G$ satisfying

$$A \otimes x = \lambda(A) \otimes x.$$

This problem was treated by several authors during the sixties, e.g. [4,7], survey of the results concerning this and similar eigenproblems can be found in [15,16].

The *ℓ-parametric eigenproblem* in max-algebra is defined similarly as follows: For an arbitrary parameter $\varepsilon \in G$ and given $A \in G(n, n)$ find $x_\varepsilon \in G(n, 1)$ and $\lambda(A_{\varepsilon^\ell}) \in G$ satisfying

$$A_{\varepsilon^\ell} \otimes x_\varepsilon = \lambda(A_{\varepsilon^\ell}) \otimes x_\varepsilon.$$

The symbol $D_A = (V, E)$ stands for a complete, arc-weighted digraph associated with A. The node set of D_A is N, and the weight of any arc (i, j) is a_{ij}. Throughout the paper, by a cycle in the digraph we mean an elementary cycle or a loop, and by path we mean a nontrivial elementary path, i.e. an elementary path containing at least one arc. Evidently, we will use the same notation, as well as the concept of weight, both for cycles and cyclic permutations. A cycle $\sigma \in C_n$ is *optimal*, if $\mu_A(\sigma) = \lambda(A)$, a node in D_A is called an *eigennode* if it is contained in at least one optimal cycle; E_A stands for the set of all eigennodes in D_A.

Theorem 2.1 (*Cuninghame-Green [5]*). *Each square matrix has at most one eigenvalue. If G is radicable, then every square matrix A has exactly one eigenvalue* (*denoted as $\lambda(A)$ in what follows*). *This unique eigenvalue is equal to the maximal average weight of cycles in D_A.*

Theorem 2.2 (*Cuninghame-Green [5]*). *Let G be radicable, $A \in G(n, n)$ and $\alpha \in G$. Then*

$$\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A).$$

The problem of finding the eigenvalue $\lambda(A)$ is also called the *maximum cycle mean problem* and it has been studied by several authors [1,2,4–7,9,10,12–14]. Various algorithms for solving this problem are known, that of Karp [10] having the best worst-case performance $O(n^3)$ and Howard's algorithm [3] of unproved computational complexity showing excellent algorithmic performance. For $B \in G(n,n)$ we denote by $\Delta(B)$ the matrix $B \oplus B^2 \oplus \ldots \oplus B^n$ where B^s stands for the s-fold iterated product $B \otimes B \otimes \ldots \otimes B$.

Let $A_\lambda = \lambda(A)^{-1} \otimes A$. (The upper index -1 denotes the inverse element of $\lambda(A)$ in the sense of the group operation $\otimes$). It is shown in [5] that the matrix $\Delta(A_\lambda)$ contains at least one column, the diagonal element of which is e. Every such column is an eigenvector of the matrix A, it is called a *fundamental eigenvector* of the matrix A. The set of all fundamental eigenvectors will be denoted by F_A and its cardinality is denoted by $q = |F_A|$. We say that $x, y \in F_A$ are equivalent if $x = \alpha \otimes y$ for some $\alpha \in G$. In what follows $s(A)$ denotes the set of all eigenvectors of A, so called *eigenspace* of A.

Theorem 2.3 (*Cuninghame-Green [5]*). *Let $A \in G(n,n)$. Then*

$$s(A) = \left\{ \sum_{i=1}^{q} \alpha_i \otimes g_i ; \alpha_i \in G, g_i \in F_A, \quad i = 1, 2, \ldots, q \right\}.$$

It follows from the definition of equivalent fundamental eigenvectors that the set F_A in Theorem 2.3 can be replaced by any maximal set F'_A of fundamental eigenvectors such that no two of them are equivalent. Every such set F'_A will be called a complete set of generators (of the eigenspace).

Theorem 2.4 (*Cuninghame-Green [5]*). *Let $g_1, g_2, \ldots, g_n$ denote the columns of the matrix $\Delta(A_\lambda)$. Then*

(i) *$j \in E_A$ if and only if $g_j \in F_A$;*
(ii) *g_i, g_j are equivalent members of F_A if and only if the eigennodes i, j are contained in a common optimal cycle.*

Let be $\Delta(A_\lambda) = (\xi_{ij})$. It follows from the definition of $\Delta(A_\lambda)$ that ξ_{ij} is the weight of the heaviest path from i to j in D_A. Hence, $\Delta(A_\lambda)$ can be computed in $O(n^3)$ operations using the Floyd–Warshall algorithm [11]. By trivial search and comparisons one can then find a complete set of fundamental eigenvectors among the columns of $\Delta(A_\lambda)$, using at most $O(n^3)$ operations.

The next assertion follows straightforwardly from the definition of $\Delta(A_\lambda)$.

Theorem 2.5. *Let $\varepsilon \in G$, $A \in G(n,n)$ and $D = \mathrm{diag}\{\varepsilon, \ldots, \varepsilon\}$. Then*

$$\Delta(A_\lambda) = \Delta((A \otimes D)_\lambda).$$

3. ℓ-Parametric eigenvalues

The aim of this section is to investigate the above *ℓ-parametric eigenproblem* over *max-plus algebra* for $A_{\varepsilon^\ell} = A \otimes D$, where A is a given matrix and $D = \mathrm{diag}(d_1, \ldots, d_\ell, 0, \ldots, 0)$ with respect to $\lambda(A_{\varepsilon^\ell})$. W.o.l.g. (Theorem 2.2) we will deal with case $G = R$ and with a given matrix A having $\lambda(A) = 0$. Suppose that a given matrix A has the following block diagonal form

$$A = \begin{pmatrix} B & . & \\ . & . & . \\ & . & C \end{pmatrix},$$

where B and C are $\ell \times \ell$ and $(n-\ell) \times (n-\ell)$ square submatrices of A, respectively. The next theorem describes very easy provable properties and the bounds of $\lambda(A_{\varepsilon^\ell})$.

Theorem 3.1.

(i) *If* $\lambda(A) = \lambda(B)$ *and* $\varepsilon \geqslant 0$ *then* $\lambda(A_{\varepsilon^\ell}) = \lambda(B) + \varepsilon$.
(ii) $\max(\lambda(B) + \varepsilon; \lambda(C)) \leqslant \lambda(A_{\varepsilon^\ell})$.

For a given matrix $A = (a_{kl}) \in G(n, n)$, $i \in N$, a cyclic permutation $\sigma = (i_1, \ldots, i_s)$, $|\{i_1, i_2, \ldots, i_s\} \bigcap \{1, 2, \ldots, \ell\}| = k$ is denoted by

$$m_s^k = \max_{\sigma \in C_n^k} \mu_A(\sigma) \left(= \max_{\sigma \in C_n^k} \left\{ \frac{a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_s i_1}}{s} \right\} \right),$$

where $C_n^k \subset C_n$ is the set of all cyclic permutations defined on subsets of N containing just k elements from $\{1, \ldots, \ell\}$.

Before the formulation of main result of this chapter we introduce a property of $\lambda(A_{\varepsilon^\ell})$.

Theorem 3.2. *If* $\varepsilon \leqslant M_n = \min_{k \leqslant i} \{(\lambda(C) - m_i^k)\frac{i}{k}\}$ *then* $\lambda(A_{\varepsilon^\ell}) = \lambda(C)$.

Proof. Suppose $\varepsilon \leqslant \min_{k \leqslant i} \{(\lambda(C) - m_i^k)\frac{i}{k}\}$ i.e.

$$(\forall i, k)(k \leqslant i)\varepsilon \leqslant (\lambda(C) - m_i^k)\frac{i}{k} \Leftrightarrow (\forall i, k)(k \leqslant i)\left(\varepsilon\frac{k}{i} + m_i^k\right) \leqslant \lambda(C)$$
$$\Leftrightarrow \lambda(A_{\varepsilon^\ell}) = \lambda(C). \qquad \square$$

Denote the following functions by:

$$m_s^k(\varepsilon) = m_s^k + \varepsilon\frac{k}{s}, \quad f(\varepsilon) = \max_{1 \leqslant k \leqslant s \leqslant n} m_s^k(\varepsilon)$$

which in *max-algebra* have the form

$$m_s^k(\varepsilon) = m_s^k \otimes \varepsilon^{(\frac{k}{s})}, \quad f(\varepsilon) = \sum_{k \leqslant s}^{\oplus} m_s^k(\varepsilon).$$

An expression of the form $\sum^{\oplus}_{k\leqslant s} m_s^k(\varepsilon)$ will be called a *maxpolynomial* [6]. In a *maxpolynomial* $f(\varepsilon)$ a term $m_s^k \otimes \varepsilon^{(\frac{k}{s})}$ will be called *inessential* if for all ε

$$m_s^k \otimes \varepsilon^{(\frac{k}{s})} \leqslant \sum^{\oplus} m_j^i(\varepsilon),$$

where the summation is taken over all pairs (i, j) satisfying $1 \leqslant i \leqslant j \leqslant n$ and $(i, j) \neq (k, s)$. Evidently an inessential term may simply be cancelled from a maxpolynomial formula without changing the function defined by the formula. All other terms will be called *essential*. The function $f(\varepsilon)$ is the upper envelope of all terms $m_s^k(\varepsilon)$. As ε increases through finite values $f(\varepsilon)$ changes only when ε passes through a corner. The essential terms $m_s^k(\varepsilon)$ and corners of the function may be determined in a linear-time procedure, as explained in [6]. To compute the corners we denote

$$L_s^k = \max_{kv>rs, r\leqslant v} \frac{vs}{kv-rs}(m_v^r - m_s^k), \quad U_s^k = \min_{kv<rs, r\leqslant v} \frac{vs}{kv-rs}(m_v^r - m_s^k)$$

for $s, k \in N$, whereby $\max \phi = -\infty$ and $\min \phi = +\infty$ are given by a definition. The breakpoints L_s^k and U_s^k for $s, k \in N$ produce the intervals which cover the axis in order that the function $f(\varepsilon)$ was specified everywhere. Now, we can formulate the following assertion which is clear from the definitions of L_s^k and U_s^k.

Theorem 3.3. $\bigcup_{k,s}[L_s^k, U_s^k] = R \cup \{-\infty, \infty\}$.

Theorem 3.4. *Let* $\varepsilon \in [L_s^k, U_s^k]$ *and* $(\forall r, v)(r \leqslant v)[(kv{=}rs) \Rightarrow (m_s^k \geqslant m_v^r)]$. *Then* $\lambda(A_{\varepsilon^\ell}) = m_s^k + \frac{k}{s}\varepsilon$.

Proof. Suppose that $\varepsilon \in [L_s^k, U_s^k]$, i.e.

$$L_s^k \leqslant \varepsilon \Leftrightarrow [(\forall r, v)(r \leqslant v)(kv > rs)\frac{vs}{kv-rs}(m_v^r - m_s^k) \leqslant \varepsilon]$$

$$\varepsilon \leqslant U_s^k \Leftrightarrow (\forall r, v)(r \leqslant v)(kv < rs)[\varepsilon \leqslant \frac{vs}{kv-rs}(m_v^r - m_s^k)].$$

Now we shall consider three possibilities.

1. $(kv = rs) \Rightarrow (m_s^k \geqslant m_v^r)$ by the assumption.
2. For $kv > rs$ we have
$$\varepsilon\frac{kv-rs}{vs} \geqslant m_v^r - m_s^k \Leftrightarrow \varepsilon\left(\frac{k}{s} - \frac{r}{v}\right) \geqslant m_v^r - m_s^k \Leftrightarrow m_s^k + \varepsilon\frac{k}{s} \geqslant m_v^r + \varepsilon\frac{r}{v}.$$
3. For $kv < rs$ we have
$$\varepsilon\frac{kv-rs}{vs} \geqslant m_v^r - m_s^k \Leftrightarrow \varepsilon\left(\frac{k}{s} - \frac{r}{v}\right) \geqslant m_v^r - m_s^k \Leftrightarrow m_s^k + \varepsilon\frac{k}{s} \geqslant m_v^r + \varepsilon\frac{r}{v}$$
and the assertion follows. □

The next assertion straightforwardly follows from the last theorem.

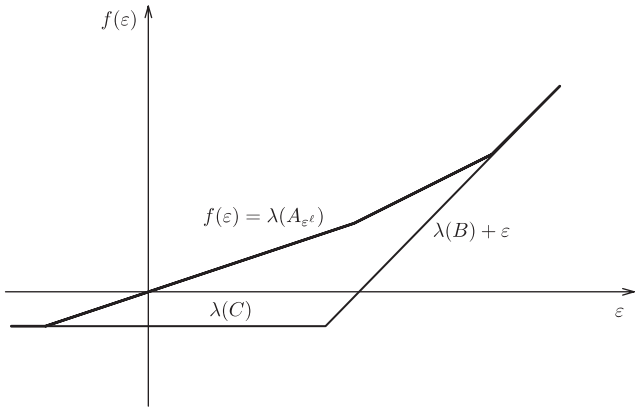

Fig. 1.

Theorem 3.5. $\lambda(A_{\varepsilon^\ell}) = \sum_{k\leqslant s}^{\oplus} m_s^k(\varepsilon) \oplus \lambda(C)$.

Theorem 3.6. *$f(\varepsilon) : R \to [\lambda(C), \infty)$ is continuous, nondecreasing, convex and piecewise-linear function with at most $\ell n - \ell^2/2$ slopes.*

Proof. The first part of the assertion straightforwardly follows from the definition of $f(\varepsilon)$ and the maximum of linear nondecreasing functions ($\frac{k}{s} \geqslant 0$) is continuous, convex and piecewise-linear function. To prove the second part we shall investigate the number different slopes of $m_s^k(\varepsilon)$. For $k \in \{1, 2, \ldots, \ell\}$ the set of feasible values for the slope of $m_s^k(\varepsilon)$ is the set $\{1, k/k + 1, \ldots, k/n\}$. The upper bound for the cardinality of the set is number $\ell n - \ell^2/2$ and the assertion follows. □

Theorems 3.4, 3.5, 3.6 allow us to construct Fig. 1 which describes the bounds and a possible graph of the function $f(\varepsilon)$ in general case.

3.1. 1-Parametric eigenvalues

To simplify the results from last section we will adapt notations which are better for computational aspects of this procedure. Again we will suppose that $\lambda(A) = 0$ (it is clear that $\lambda(C) \leqslant \lambda(A)$). For a given matrix $A = (a_{kl}) \in G(n, n),\ i \in N,\ \sigma = (1, j_2, \ldots, j_i)$ is denoted by

$$m_i^1 = \max_{\sigma \in C_n^1} \left\{ \frac{a_{1j_2} + a_{j_2 j_3} + \cdots + a_{j_i 1}}{i} \right\},$$

where $C_n^1 \subset C_n$ is the set of all cyclic permutations defined on subsets of N containing the element 1.

Table 1

ε	$(-\infty,-3]$	$[-3,6]$	$[6,10]$	$[10,\infty)$
$\lambda(A_{\varepsilon^1})$	-1	$\varepsilon/3$	$-1+\varepsilon/2$	$-6+\varepsilon$

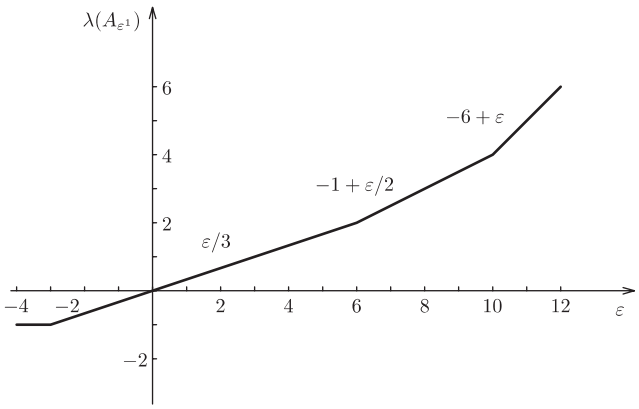

Fig. 2.

Now we can define $M_n = \min\{(\lambda(C) - m_i^1)i;\ \ i \in N\}$ and for $s \in N$ we put

$$L_s^1 = \max_{v>s} \frac{vs}{v-s}(m_v^1 - m_s^1), \quad U_s^1 = \min_{v<s} \frac{vs}{v-s}(m_v^1 - m_s^1).$$

Theorem 3.7.

(i) *If* $\varepsilon \leqslant M_n$ *then* $\lambda(A_{\varepsilon^1}) = \lambda(C)$.
(ii) *If* $\varepsilon \in [L_s^1, U_s^1]$ *then* $\lambda(A_{\varepsilon^1}) = m_s^1 + \varepsilon/s$.

Example. Let matrix A have the following form

$$A = \begin{pmatrix} -6 & 0 & -4 \\ -2 & -3 & -2 \\ 2 & 0 & -1 \end{pmatrix}.$$

Then $m_1^1 = -6,\ m_2^1 = -1,\ m_3^1 = 0$ and

$$M_3 = -3,\ [L_3^1, U_3^1] = [-3,6],\ [L_2^1, U_2^1] = [6,10],\ [L_1^1, U_1^1) = [10,\infty).$$

Using Theorem 3.4 we can split the real axis into several intervals as described in Table 1 and Fig. 2.

4. ℓ-Parametric eigenvectors

In this section we deal with an analysis of eigenvectors with respect to parameter ε. To describe the eigenspace dependent on parameter ε we shall formulate the following theorem.

Theorem 4.1.

(i) *If* $\lambda(A_{\varepsilon^\ell}) = \lambda(C)$ *then* $|F'_{A_{\varepsilon^\ell}}| \leqslant \ell + |F'_C|$.
(ii) *If* $\lambda(A_{\varepsilon^\ell}) > \lambda(C)$ *then* $|F'_{A_{\varepsilon^\ell}}| \leqslant \ell$.
(iii) *If* $\lambda(A_{\varepsilon^\ell}) < \lambda(C)$ *then* $|F'_{A_{\varepsilon^\ell}}| = |F'_C|$.

Proof. The upper bounds for the cardinality of $F'_{A_{\varepsilon^\ell}}$ follow straightforwardly from the definitions of the ℓ-parametric eigenvalue. □

To find the coordinates of the eigenvector we need to know the columns of metric matrix $\Delta((A_{\varepsilon^\ell})_\lambda) = (\xi_{ij}(\varepsilon))$. For computing it we will use the following notation. The matrix with elements $w_{i,j}^{u,p}$ which describe the weight of the heaviest path from node i to the node j of length u containing p nodes from the set $\{1, \ldots, \ell\}$ in D_A denote by $W = (w_{i,j}^{u,p})$. We can define the function $\xi_{ij}^{u,p}(\varepsilon) = w_{i,j}^{u,p} + p\varepsilon - u\lambda(A_{\varepsilon^\ell})$. Moreover, if $\varepsilon \in [L_s^k, U_s^k]$ the last formula can be rewritten as follows:

$$\xi_{ij}^{u,p}(\varepsilon) = w_{i,j}^{u,p} + p\varepsilon - u\left(m_s^k + \frac{k}{s}\varepsilon\right) = w_{i,j}^{u,p} - um_s^k + \varepsilon\left(p - \frac{uk}{s}\right)$$

or in notations of max-algebra we have $\xi_{ij}^{u,p}(\varepsilon) = (w_{i,j}^{u,p} \otimes (m_s^k)^{(-u)}) \otimes \varepsilon^{(p-uk/s)}$ and put $\xi_{ij}(\varepsilon) = \sum_{u,p}^{\oplus} \xi_{ij}^{u,p}(\varepsilon)$. The function $\xi_{ij}(\varepsilon)$ is again maxpolynomial and it is the upper envelope of all terms $\xi_{ij}^{u,p}(\varepsilon)$ whereby $\xi_{ij}(\varepsilon) : R \to R$ is continuous and piecewise-linear function. Similarly, as in the last section the essential terms and corners of $\xi_{ij}(\varepsilon)$ may be determined in a linear-time procedure [6]. To compute the corners we denote

$$L_{i,j}^{u,p} = \max_{\substack{g,r \\ g>u}} \frac{w_{i,j}^{g,r} - w_{i,j}^{u,p} + (r-p)\varepsilon}{g-u}, \quad U_{i,j}^{u,p} = \min_{\substack{g,r \\ g<u}} \frac{w_{i,j}^{g,r} - w_{i,j}^{u,p} + (r-p)\varepsilon}{g-u},$$

for $i, j, u, p \in N$, whereby $\max \phi = -\infty$ and $\min \phi = +\infty$ are given by definition. The breakpoints $L_{i,j}^{u,p}$ and $U_{i,j}^{u,p}$ for $u, p \in N$ produce the intervals which cover the axis in order that the function $\xi_{ij}(\varepsilon)$ was specified everywhere. Now, we can formulate the following assertion which is clear from the definitions of $L_{i,j}^{u,p}$ and $U_{i,j}^{u,p}$.

Theorem 4.2. *Let* $i, j \in N$ *be fixed. Then* $\bigcup_{p,u}[L_{i,j}^{u,p}, U_{i,j}^{u,p}] = R \cup \{-\infty, \infty\}$.

Theorem 4.3. *Let* $\lambda(A_{\varepsilon^\ell}) \in [L_{i,j}^{u,p}, U_{i,j}^{u,p}]$. *Then* $\xi_{i,j}(\varepsilon) = w_{i,j}^{u,p} + p\varepsilon - u\lambda(A_{\varepsilon^\ell})$.

Proof. Suppose

$$\lambda(A_{\varepsilon^\ell}) \in [L_{i,j}^{u,p}, U_{i,j}^{u,p}] \Leftrightarrow L_{i,j}^{u,p} \leqslant \lambda(A_{\varepsilon^\ell}) \leqslant U_{i,j}^{u,p} \Leftrightarrow$$

$$(\forall g, r)(g > u)\left(\frac{w_{i,j}^{g,r} - w_{i,j}^{u,p} + (r-p)\varepsilon}{g-u} \leqslant \lambda(A_{\varepsilon^\ell})\right)$$

$$(\forall g, r)(g < u)\left(\frac{w_{i,j}^{g,r} - w_{i,j}^{u,p} + (r-p)\varepsilon}{g-u} \geqslant \lambda(A_{\varepsilon^\ell})\right)$$

$$\Leftrightarrow (\forall g, r)(w_{i,j}^{u,p} + p\varepsilon - u\lambda(A_{\varepsilon^\ell}) \geqslant w_{i,j}^{g,r} + r\varepsilon - g\lambda(A_{\varepsilon^\ell})). \qquad \square$$

4.1. 1-parametric eigenvectors

From the previous section it is clear that for each $\varepsilon > M_n$ the set $F'_{A_{\varepsilon^1}}$ has cardinality equal to 1 and $1 \in E_{A_{\varepsilon^1}}$. To find the coordinates of the eigenvector we need to know the first column of metric matrix $\Delta((A_{\varepsilon^1})_\lambda) = (\xi_{ij}(\varepsilon))$. For computing it we will use the following notation.

The matrix with elements $w_{i,1}^u$ which describe the weight of the heaviest path from node i to the node 1 of length u in D_A will be denoted by $W = (w_{i,1}^u)$.

For $u \in N$ put

$$L_{i,1}^u = \max_{l>u} \frac{w_{i,1}^l - w_{i,1}^u}{l-u}, \quad U_{i,1}^u = \min_{l<u} \frac{w_{i,1}^u - w_{i,1}^l}{u-l}.$$

Theorem 4.4. *Let* $\lambda(A_{\varepsilon^1}) \in [L_{i,1}^u, U_{i,1}^u]$. *Then* $\xi_{i,1}(\varepsilon) = w_{i,1}^u + \varepsilon - u\lambda(A_{\varepsilon^1})$.

5. Computational aspects

The previous sections describe the procedure which computes all eigenvalues and corresponding eigenvectors dependent on parameter ε. To give the computational complexity of the considered procedure we will use the $O(n^3)$ Karp's and Floyd-Warshall's algorithms.

Procedure **1-Parameter**
Input: A given matrix A
Output: $\lambda(A_{\varepsilon^1})$, x_ε
1. *Compute* $\lambda(A)$, $\lambda(C)$, M_n, $[L_{i,1}^u, U_{i,1}^u]$
2. *Compute* $W = (w_{i,1}^u)$
3. *Describe* x_ε.

Theorem 5.1. *Procedure* **1-Parameter** *works correct and terminates after* $O(n^3)$ *steps.*

Proof. The first and third steps are clear. Denote $w_1^k = \begin{pmatrix} w_{1,1}^k \\ w_{2,1}^k \\ \vdots \\ w_{n,1}^k \end{pmatrix}$. The second step follows from the fact that it is possible to compute the elements of the matrix W by the formula:

$$w_1^{k+1} = A \otimes w_1^k$$

for $k = 1, \ldots, n-1$. Then each step has the best worst-case performance $O(n^3)$. □

Example-continuation.

For the computing eigenvectors we will construct the matrix W by using the formula $w_1^{k+1} = A \otimes w_1^k$. Then

$$W = \begin{pmatrix} w_{1,1}^1 & w_{1,1}^2 & w_{1,1}^3 \\ w_{2,1}^1 & w_{2,1}^2 & w_{2,1}^3 \\ w_{3,1}^1 & w_{3,1}^2 & w_{3,1}^3 \end{pmatrix} = \begin{pmatrix} -6 & -2 & 0 \\ -2 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}.$$

From Theorem 4.1 it follows that $|F'_{A_{\varepsilon^\ell}}| = |F'_C|$ for $\lambda(A_{\varepsilon^\ell}) < \lambda(C)$. We will consider the case when $\varepsilon \in [-1, +\infty)$ only. For $\varepsilon \in (-\infty, -1)$ is $\lambda(A_{\varepsilon^\ell}) = \lambda(C) = -1$ and $F'_{A_{\varepsilon^1}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Now from the matrix W we can compute intervals $[L_{i,1}^u, U_{i,1}^u]$ and $\xi_{i,1}(\varepsilon)$ as follows:

$$L_{1,1}^1 = \max_{l>1} \frac{w_{1,1}^l - w_{1,1}^1}{l-1} = \max\left\{\frac{4}{1}, \frac{6}{3}\right\} = 4, \quad U_{1,1}^1 = \min_{l<1} \frac{w_{1,1}^l - w_{1,1}^1}{l-1} = +\infty.$$

For $\lambda(A_{\varepsilon^1}) \in [4, +\infty)$ is $\xi_{1,1}(\varepsilon) = -6 + \varepsilon - \lambda(A_{\varepsilon^1})$.

$$L_{1,1}^2 = \max_{l>2} \frac{w_{1,1}^l - w_{1,1}^2}{l-2} = 2, \quad U_{1,1}^2 = \min_{l<2} \frac{w_{1,1}^l - w_{1,1}^2}{l-2} = 4.$$

For $\lambda(A_{\varepsilon^1}) \in [2, 4]$ is $\xi_{1,1}(\varepsilon) = -2 + \varepsilon - 2\lambda(A_{\varepsilon^1})$.

$$L_{1,1}^3 = \max_{l>3} \frac{w_{1,1}^l - w_{1,1}^3}{l-3} = -\infty, \quad U_{1,1}^3 = \min_{l<3} \frac{w_{1,1}^l - w_{1,1}^3}{l-3} = \min\left\{\frac{-6}{-2}, \frac{2}{1}\right\} = 2.$$

For $\lambda(A_{\varepsilon^1}) \in [-1, 2]$ is $\xi_{1,1}(\varepsilon) = 0 + \varepsilon - 3\lambda(A_{\varepsilon^1})$.

$$L_{2,1}^1 = \max_{l>1} \frac{w_{2,1}^l - w_{2,1}^1}{l-1} = \max\left\{\frac{2}{1}, \frac{1}{2}\right\} = 2, \quad U_{2,1}^1 = \min_{l<1} \frac{w_{2,1}^l - w_{2,1}^1}{l-1} = +\infty.$$

For $\lambda(A_{\varepsilon^1}) \in [2, +\infty)$ is $\xi_{2,1}(\varepsilon) = -2 + \varepsilon - \lambda(A_{\varepsilon^1})$.

$$L_{2,1}^2 = \max_{l>2} \frac{w_{2,1}^l - w_{2,1}^2}{l-2} = -1, \quad U_{2,1}^2 = \min_{l<2} \frac{w_{2,1}^l - w_{2,1}^2}{l-2} = 2.$$

Table 2

$\lambda(A_{\varepsilon^1})$	$(-\infty,-1]$	$[-1,2]$	$[2,4]$	$[4,\infty)$
x_ε	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0+\varepsilon-3\lambda(A_{\varepsilon^1})\\0+\varepsilon-2\lambda(A_{\varepsilon^1})\\2+\varepsilon-\lambda(A_{\varepsilon^1})\end{pmatrix}$	$\begin{pmatrix}-2+\varepsilon-2\lambda(A_{\varepsilon^1})\\-2+\varepsilon-\lambda(A_{\varepsilon^1})\\2+\varepsilon-\lambda(A_{\varepsilon^1})\end{pmatrix}$	$\begin{pmatrix}-6+\varepsilon-\lambda(A_{\varepsilon^1})\\-2+\varepsilon-\lambda(A_{\varepsilon^1})\\2+\varepsilon-\lambda(A_{\varepsilon^1})\end{pmatrix}$

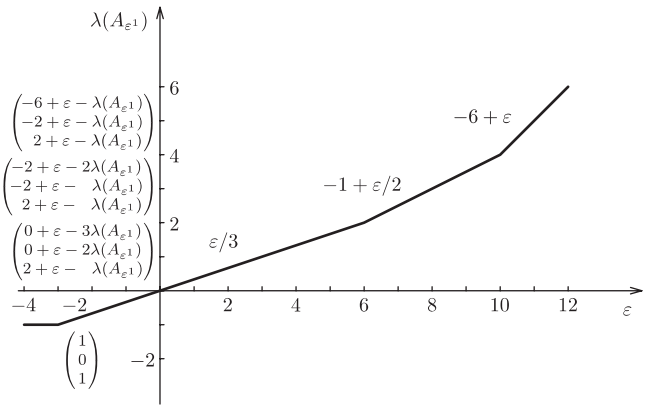

Fig. 3.

For $\lambda(A_{\varepsilon^1}) \in [-1,2]$ is $\xi_{2,1}(\varepsilon) = 0+\varepsilon-2\lambda(A_{\varepsilon^1})$.

$$L^1_{3,1} = \max_{l>1} \frac{w^l_{3,1} - w^1_{3,1}}{l-1} = \max\left\{\frac{-1}{1}, \frac{-2}{2}\right\} = -1,$$

$$U^1_{3,1} = \min_{l<1} \frac{w^l_{3,1} - w^3_{3,1}}{l-1} = +\infty.$$

For $\lambda(A_{\varepsilon^1}) \in [-1,\infty)$ is $\xi_{3,1}(\varepsilon) = 2+\varepsilon-\lambda(A_{\varepsilon^1})$.

Now we can summarize all obtained results into Table 2 and Fig. 3.

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Appendix

To solve the ℓ-parametric eigenproblem in the general case means to suggest the procedure for computing intervals for ℓ-parametric eigenvalue and coordinates $\xi_{i,1}(\varepsilon)$

of ℓ-parametric eigenvectors. For $\ell \geqslant 2$ this problem seems to be hard since we do not know an efficient algorithm for computation of m_i^2 and $w_{i,j}^{u,2}$ in the general case.

The *general ℓ-parametric eigenproblem* can be defined as follows:

For arbitrary parameters $\varepsilon_i \in G$, $i \in \{1, 2, \ldots, \ell\}$ and given $A \in G(n, n)$ find $x_{\varepsilon_1,\ldots,\varepsilon_\ell} \in G(n, 1)$ and $\lambda(A_{\varepsilon_1,\ldots,\varepsilon_\ell}) \in G$ satisfying

$$A_{\varepsilon_1,\ldots,\varepsilon_\ell} \otimes x_{\varepsilon_1,\ldots,\varepsilon_\ell} = \lambda(A_{\varepsilon_1,\ldots,\varepsilon_\ell}) \otimes x_{\varepsilon_1,\ldots,\varepsilon_\ell}.$$

One possible way to solve the *general ℓ-parametric eigenproblem* is to use the mathematical model and procedure **1-Parameter**. For the model, we will suppose w.l.o.g. that ε_1 is a parameter and $\varepsilon_2, \ldots, \varepsilon_\ell$ are fixed values given by experts. By using procedure **1-Parameter** on the model can be obtained a first approximation of a solution of the problem. The second one can be obtained by using the other combination of parameter and fixed values. The process allows us to find a desired solution.

As we assert above the special case of *general ℓ-parametric eigenproblem* ($\varepsilon_1 = \cdots = \varepsilon_\ell = \varepsilon$) is hard. In spite of the fact, we can suggest a possible method for looking for a solution of the problem.

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